AN ANALYTICAL TREATMENT OF A CLASS OF PLANE PLASTIC STRAIN PROBLEMS

by

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1. *Introduction.*

Nottrot and Timman [1] describe the solution to the plane elasto-plastic problem in the region around a hole of circular shape. When one tries to extend the method to other than circular shapes, one meets first of all the question of the determination of the stresses in the plastic region adjacent to the hole. Apart from this question, the plane elasto-plastic problem can be solved in roughly the same way as was done by Nottrot and Timman. Therefore it seems superfluous to describe more in this paper than the solution to the plane plastic strain problem in the region around hole of general shape.

In general, such a problem can be solved by numerical or graphical methods, making use of Kötter's equations. The way the elasto-plastic problem is solved, however, requires an analytical expression. One should remember that the boundary between the plastic and elastic regions is found by a trial-and-error procedure; this means of course that the boundary is shifted in each step by a greater or lesser amount. Only an analytical expression yields the required values without loss of accuracy.

The above mentioned equations of Kötter, though well adapted for graphical or numerical methods, do not lend themselves to an analytical treatment because they are non-linear. It was suggested to me by prof. Timman to apply a method, also indicated by Sokolovsky [2, 3] and Geiringer [4], to obtain linear equations by a conversion of the problem. Then the Cartesian coordinates x and y of the physical plane are considered as the unknown functions of the isotropic stress σ, and θ, which is the angle between the major principal stress and the x-axis. In the following paper the equations are derived for the plane plastic strain problem, where yielding is governed by a generalised Mohr-Coulomb condition.

The theory is based on the same assumptions as Kötter's theory, namely, the absence of time effects, and the absence of influence of the intermediate principal stress. One further restriction must be made: all body forces, like weight and excess pore pressure gradient, are left out of consideration.

The procedure in the case of the analytical treatment is quite different from the numerical or the graphical method. While in these latter cases the computation starts from the given boundary condition, in the analytical treatment particular solutions are first sought, and these ultimately are combined to satisfy some given boundary condition.

The linear differential equations, derived in sec. 2 below, are solved in the next sections for a medium such as a sand, which follows the normal Mohr-Coulomb yield condition. The same procedure can, however, be applied to a medium not possessing internal friction.

2. *Plane Plastic Strain.*

Consider the case of plane plastic strain, where the plasticity condition is assumed to be given in the following form: the radius τ of Mohr's circle is a given function of the isotropic stress σ,

$$\tau = \left[\tfrac{1}{4}(\sigma_y - \sigma_x)^2 + \tau_{xy}^2\right]^{\frac{1}{2}} = \tau(\sigma), \tag{1}$$

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where $\sigma = \frac{1}{2}(\sigma_x + \sigma_y)$. **(2)**

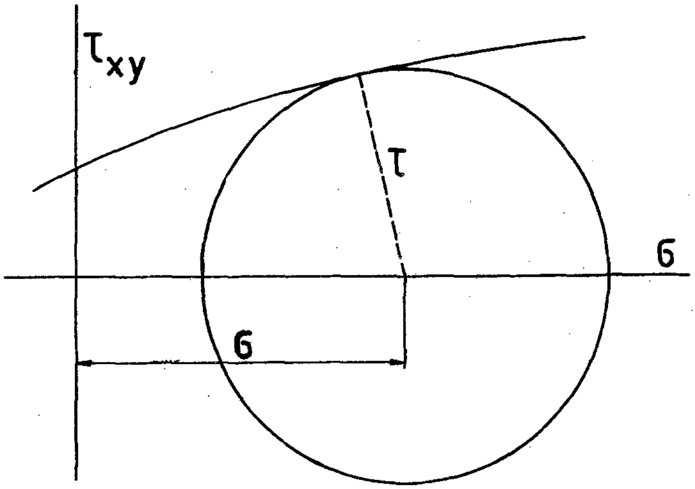

Fig. 1.

We introduce the direction of the major principal stress, **θ**, in the following way:

$$\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\} = \sigma \pm \tau(\sigma)\cos 2\theta, \qquad \tau_{xy} = \tau(\sigma)\sin 2\theta. \tag{3}$$

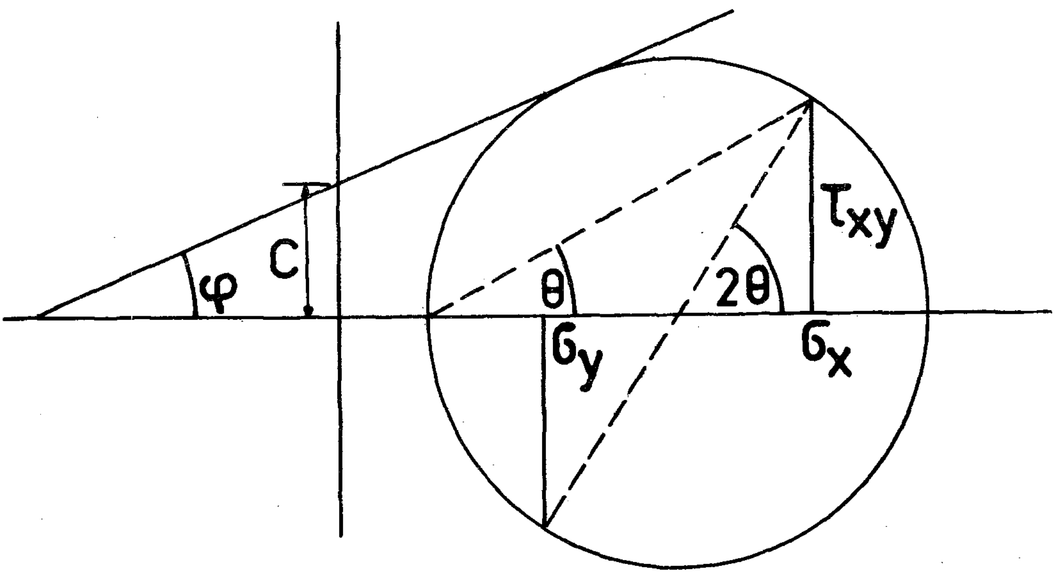

Fig. 2.

In the absence of body forces the equations of equilibrium take the form

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = (1 + \tau'\cos 2\theta)\frac{\partial\sigma}{\partial x} - 2\tau\sin 2\theta\frac{\partial\theta}{\partial x} +$$

$$+ \tau'\sin 2\theta\frac{\partial\sigma}{\partial y} + 2\tau\cos 2\theta\frac{\partial\theta}{\partial y} = 0,$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} = \tau' \sin 2\theta \frac{\partial\sigma}{\partial x} + 2\tau \cos 2\theta \frac{\partial\theta}{\partial x} + $$
$$+ (1 - \tau' \cos 2\theta) \frac{\partial\sigma}{\partial y} + 2\tau \sin 2\theta \frac{\partial\theta}{\partial y} = 0,$$

where $\frac{d\tau}{d\sigma} = \tau'$.

These equations can be simplified by a linear combination, where the factors by which the equations are multiplied, are $+\cos 2\theta$ and $+\sin 2\theta$ on the one hand, and $+\sin 2\theta$ and $-\cos 2\theta$ on the other.

We find

$$\begin{aligned} &(\cos 2\theta + \tau') \frac{\partial\sigma}{\partial x} + \sin 2\theta \frac{\partial\sigma}{\partial y} + 2\tau \frac{\partial\theta}{\partial y} = 0, \\ &\sin 2\theta \frac{\partial\sigma}{\partial x} - 2\tau \frac{\partial\theta}{\partial x} + (\tau' - \cos 2\theta) \frac{\partial\sigma}{\partial y} = 0. \end{aligned} \tag{4}$$

The unknown functions σ and θ appear in the coefficients, so the equations (4) are clearly non-linear. The coefficients, on the other hand, do not depend on x and y; this is the reason that it is fruitful to consider x and y as the unknown functions, and σ and θ as the independent variables. To find the differential equations of the reverse problem, one needs the Jacobian transformation determinant Δ:

$$\Delta = \begin{vmatrix} \frac{\partial\sigma}{\partial x} & \frac{\partial\sigma}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{vmatrix},$$

so that $\frac{\partial\sigma}{\partial x} = \Delta \frac{\partial y}{\partial\theta}$, $\frac{\partial\sigma}{\partial y} = -\Delta \frac{\partial x}{\partial\theta}$, $\frac{\partial\theta}{\partial x} = -\Delta \frac{\partial y}{\partial\sigma}$, $\frac{\partial\theta}{\partial y} = \Delta \frac{\partial x}{\partial\sigma}$.

Assuming $\Delta \neq 0$, equation (4) can be transformed into

$$\begin{aligned} &(\tau' + \cos 2\theta) \frac{\partial y}{\partial\theta} - \sin 2\theta \frac{\partial x}{\partial\theta} + 2\tau \frac{\partial x}{\partial\sigma} = 0, \\ &\sin 2\theta \frac{\partial y}{\partial\theta} + 2\tau \frac{\partial y}{\partial\sigma} + (\cos 2\theta - \tau') \frac{\partial x}{\partial\theta} = 0. \end{aligned} \tag{5}$$

These are indeed linear equations, for every choice of the function $\tau(\sigma)$. Let us now choose $\tau(\sigma)$ as it is for a granular material

$$\tau(\sigma) = \sin\varphi(\sigma + c \operatorname{cotg}\varphi), \tag{6}$$

where c is the cohesion, and φ is the angle of internal friction (see fig. 2). For compactness we shall write

$$\rho = \sin\varphi,$$

and
$$s = (\sigma + c \operatorname{cotg}\varphi)/\sigma_0, \tag{7}$$

σ_0 being an arbitrary constant, and it follows that

$$\tau(\sigma) = \rho\sigma_0 s, \quad \tau' = \rho, \quad \frac{\partial}{\partial\sigma} = \sigma_0^{-1} \frac{\partial}{\partial s}$$

Hence the equations (5), written in terms of the new variables, and with the above yield criterion, become

$$(\cos 2\theta + \rho)\frac{\partial y}{\partial \theta} - \sin 2\theta \frac{\partial x}{\partial \theta} + 2\rho s \frac{\partial x}{\partial s} = 0,$$
$$\sin 2\theta \frac{\partial y}{\partial \theta} + 2\rho s \frac{\partial y}{\partial s} + (\cos 2\theta - \rho)\frac{\partial x}{\partial \theta} = 0. \qquad (8)$$

These equations can be solved by the separation of variables. One sees that in (8), the partial derivatives $\partial x/\partial s$ and $\partial y/\partial s$ are both multiplied by a factor s. The separation can thus be effected by putting

$$\left.\begin{aligned} x &= s^{\lambda}X(\theta) \\ y &= s^{\lambda}Y(\theta) \end{aligned}\right\} \qquad (9)$$

Then all coefficients can be divided by s^{λ}, so one finds ordinary differential equations for X and Y. The complete solution of the partial differential equations will be a linear combination of the particular solutions, found from these ordinary linear differential equations. It is also seen that all coefficients are trigonometrical functions of θ, and we shall therefore write

$$z = e^{i\theta} \qquad (10)$$

and so (8) now becomes, omitting the factor s^{λ},

$$\tfrac{1}{2}\left[(z^2 + z^{-2} + 2\rho)iz\frac{dY}{dz} - (z^2 - z^{-2})z\frac{dX}{dz} + 4\lambda\rho X\right] = 0 \qquad (11)'$$

$$\tfrac{1}{2}\left[(z^2 - z^{-2})z\frac{dY}{dz} + 4\lambda\rho Y + (z^2 + z^{-2} - 2\rho)iz\frac{dX}{dz}\right] = 0 \qquad (11)''$$

The expressions for X + iY and X - iY will be seen to be very simple, much simpler indeed than those for X and Y alone. These expressions are found by a suitable linear combination of (11)' and (11)''.

If (11)' and (11)'' are multiplied by $(1 - \rho z^2)$ and $i(1 + \rho z^2)$ respectively, and then added, there results

$$X + iY = \rho z^2(X - iY) - \{(\rho^2 - 1)/2\lambda\rho\}z^3\frac{d(X - iY)}{dz} \qquad (12)$$

If (11)' and (11)'' are multiplied by $1 - \rho z^{-2}$ and $i(1 + \rho z^{-2})$ respectively, and then substracted, there results

$$X - iY = \rho z^{-2}(X + iY) + \{(\rho^2 - 1)/2\lambda\rho\}z^{-1}\frac{d(X + iY)}{dz}. \qquad (13)$$

By elimination of X + iY and X - iY respectively, we obtain

$$4\lambda^2\rho^2(X + iY) + 4\lambda\rho^2(X + iY) - (\rho^2 - 1)z^3\frac{d}{dz}\left[z^{-1}\frac{d(X + iY)}{dz}\right] = 0,$$
$$4\lambda^2\rho^2(X - iY) + 4\lambda\rho^2(X - iY) - (\rho^2 - 1)z^{-1}\frac{d}{dz}\left[z^3\frac{d(X - iY)}{dz}\right] = 0. \qquad (14)$$

Powers of z are solutions to these equations, because in every term the powers of z in the coefficients are in balance (d/dz being equivalent to z^{-1}). These solutions can therefore be written as

$$X + iY = C_m z^{1+m}, \qquad X - iY = D_m z^{-1+m}. \tag{15}$$

The value of m is still dependent on λ; or, conversely, λ can be found when m is given; by substitution into (14) it is found that

$$\lambda_m^2 + \lambda_m + \left[(1-\rho^2)/4\rho^2\right](m^2 - 1) = 0. \tag{16}$$

The relation between C_m and D_m can also be established from these equations, for from (13) it follows that

$$D_m = \{\rho - (1+m)(1-\rho^2)/2\lambda_m\rho\}C_m = \\ = \rho\{1 + (2\lambda_m + 2)/(m-1)\}C_m. \tag{17}$$

We will limit ourselves to the case where x and y are periodic functions of θ, with period 2π (see also sec. 4). This means that m must be an integer, since we assumed $z = e^{i\theta}$.

Therefore the particular solutions corresponding to $m = 0, \pm 1, \pm 2, \ldots$ have to be studied. For each m two values of λ are found from (16):

$$\left.\begin{matrix}\lambda_m \\ \lambda'_m\end{matrix}\right\} = -\tfrac{1}{2} \pm \tfrac{1}{2}\left[1 - m^2(1-\rho^2)\right]^{\frac{1}{2}}/\rho; \tag{18}$$

λ_m and λ'_m are real when $m = 0$ or $m = \pm 1$, and complex when $m = \pm 2, \pm 3, \ldots$ if we limit ourselves to: $0 < \rho < \frac{1}{2}\sqrt{3}$ or $0 < \varphi < \frac{1}{3}\pi$, but this limitation is of course by no means essential.

We have now found particular solutions of the form $C_m s^{\lambda_m} z^{1+m}$, as (15) and (9) show. Because the superposition principle holds for the differential equations (5), the particular solutions can be added to yield the complete solution

$$x + iy = \sum_{m=-\infty}^{+\infty}\left[C_m s^{\lambda_m} z^{1+m} + C'_m s^{\lambda'_m} z^{1+m}\right]. \tag{19}$$

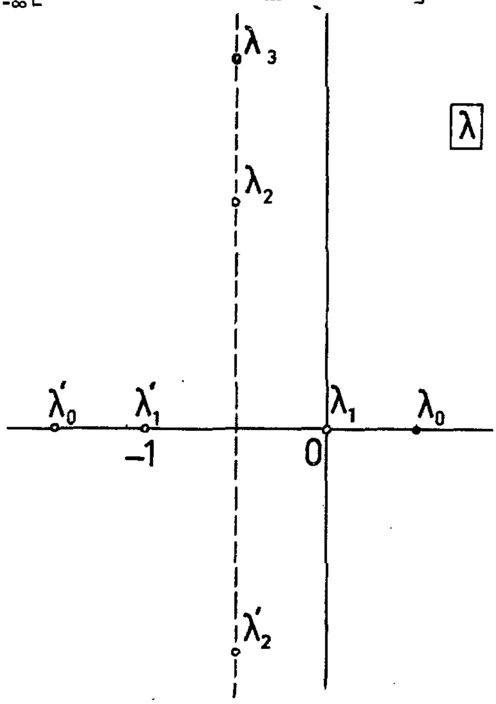

Fig. 3. Representation of λ_m and λ'_m in the complex plane.

3. *The Particular Solutions.*

In this section the particular solutions are to be studied, in the order $m = 0$, $m = \pm 1, \pm 2$ and so forth. This ordering is important, because it is clear from (18) that $\lambda_m = \lambda_{-m}$ and $\lambda'_m = \lambda'_{-m}$.

The principal solutions are those corresponding to $m = 0$:

$$\lambda_0 = -\tfrac{1}{2} + \tfrac{1}{2}\rho^{-1}, \text{ so from (17): } D_0 = -C_0.$$

The solution (15) is in this case

$$\left.\begin{aligned} X + iY &= C_0 z \\ X - iY &= D_0 z^{-1} \end{aligned}\right\}$$

so that, according to (9)

$$\left.\begin{aligned} x + iy &= C_0 s^{\lambda_0} z \\ x - iy &= D_0 s^{\lambda_0} z^{-1} \end{aligned}\right\}$$

The functions x and y themselves must be real, otherwise they would not represent physical reality. Therefore $x - iy$ must be the complex conjugate of $x + iy$. The factor s^{λ_0} is real, z and z^{-1} are already complex conjugates, so C_0 and D_0 must also be conjugates. If we denote the conjugate of C_0 by $\bar{C}_0$, we see that

$$D_0 = \bar{C}_0 .$$

It was already seen that $D_0 = -C_0$, so $C_0 = -\bar{C}_0$, which can only be true when C_0 is purely imaginary.

This solution

$$\left.\begin{aligned} x &= +iC_0 s^{[(1-\rho)/2\rho]} \sin\theta \\ y &= -iC_0 s^{[(1-\rho)/2\rho]} \cos\theta \end{aligned}\right\} \tag{20}$$

corresponds to the solution of the stress distribution around a circular hole. Introducing the radial coordinate

$$r = (x^2 + y^2)^{\frac{1}{2}} ,$$

(20) can be modified to

$$r = |C_0|\, s^{[(1-\rho)/2\rho]} ,$$

therefore

$$s = s_0\, r^{[2\rho/(1-\rho)]} ,$$

where s_0 is a constant whose value follows from the boundary condition. Terzaghi [5] finds in this case

$$\sigma_r = \sigma_{r_0} (r/r_0)^{N_\varphi - 1}$$

where σ_r corresponds to s:

$$N_\varphi = \mathrm{tg}^2(45^\circ + \tfrac{1}{2}\varphi) = \frac{\sin^2(\frac{1}{4}\pi + \frac{1}{2}\varphi)}{\cos^2(\frac{1}{4}\pi + \frac{1}{2}\varphi)} = \frac{\frac{1}{2} - \frac{1}{2}\cos(\frac{1}{2}\pi + \varphi)}{\frac{1}{2} + \frac{1}{2}\cos(\frac{1}{2}\pi + \varphi)} = \frac{1 + \rho}{1 - \rho},$$

so $N_{\varphi} - 1 = 2\rho/(1 - \rho)$, the exponents are indeed identical.

The second case, corresponding to $m = 0$:

$$\lambda'_0 = -\tfrac{1}{2} - \tfrac{1}{2}\rho^{-1}, \text{ so (17) yields } D'_0 = C'_0 .$$

Again as in the preceding case

$$x - iy = \overline{x + iy}, \text{ or } D'_0 s^{\lambda'_0} z^{-1} = \overline{C'_0 s^{\lambda'_0} z} = \overline{C'_0} s^{\lambda'_0} z^{-1} .$$

$D'_0 = \overline{C'_0} = C'_0$, so C'_0 is real,

$$\begin{aligned} x &= C'_0 s^{-[(1+\rho)/2\rho]} \cos\theta \\ y &= C'_0 s^{-[(1+\rho)/2\rho]} \sin\theta \end{aligned} \tag{21}$$

Both solutions (20) and (21) represent the plastic stress distribution around a circular hole. Solution (20) represents the case when the stresses increase without limit for large values of x and y ($s - \infty$). In the other case (21) s reduces to zero, $\tau \to 0$ and σ approaches $-c \operatorname{cotg} \varphi$ at a great distance from the hole. In the case of a hole of general shape, there are the same two possibilities for the behaviour at infinity. Since, however, the values of λ, corresponding to $m = 0$, happen to be the ones with the greatest and the smallest real parts (see fig. 3), either solution (20) or (21) will dominate. Therefore at a great distance from the hole a stress distribution resembling a circular one, is found.

The next solutions, corresponding to $m = \pm 1, \pm 2, \ldots$ can be considered as disturbances of the principal solutions already found. These disturbances appear when the shape of the hole is not a circle, or when the load applied to the boundary is not homogeneously distributed.

With $\lambda_1 = 0$, from (14) we see that $\frac{d(X+iY)}{dz} = 0$, $\frac{d(X-iY)}{dz} = 0$, so that

$$X + iY = \text{const} = C_{-1}$$

and $$X - iY = \text{const} = D_{-1}$$

The factor $s^{\lambda_1} = 1$, so that

$$\begin{aligned} x + iy &= C_{-1} , \\ C_1 &= D_1 = 0. \end{aligned} \tag{22}$$

This solution, $x = \text{const}$, $y = \text{const}$, represents merely a translation of the coordinate axes, when added to other solutions.

With $\lambda'_1 = -1$, then $D'_1 = \rho^{-1} C'_1$
$D'_{-1} = \rho C'_{-1}$ from (17), and so

$$\begin{aligned} x + iy &= s^{-1}\left[C'_1 z^2 + C'_{-1}\right] , \\ x - iy &= s^{-1}\left[\rho^{-1} C'_1 + \rho C'_{-1} z^{-2}\right] . \end{aligned}$$

Since x and y must be real, $C'_{-1} = \rho^{-1}\overline{C'_1}$;

$$x + iy = s^{-1}\left[\rho \overline{C'_{-1}} e^{2i\theta} + C'_{-1}\right]. \tag{23}$$

Writing $C'_{-1} = Ce^{-i\alpha}$, then

$$\begin{aligned} x &= Cs^{-1}\left[\rho \cos(2\theta + \alpha) + \cos\alpha\right], \\ y &= Cs^{-1}\left[\rho \sin(2\theta + \alpha) - \sin\alpha\right]. \end{aligned}$$

As will be proved in sec. 5, *iii*, this term of the complete solution accounts for the resultant force of the external forces, exerted on the boundary curve. The resultant force will be shown to be $4\pi\rho\sigma_0 C'_1$.

For $m = \pm 2, \pm 3, \ldots$ λ_m and λ'_m are complex; let us write

$$\lambda_m, \lambda'_m = -\tfrac{1}{2} \pm i\nu_m, \tag{24}$$

where $\nu_m = \tfrac{1}{2}\rho^{-1}\left[(1-\rho^2)m^2 - 1\right]^{\frac{1}{2}}$.

In order to obtain real values for x and y, the solutions

$$\left.\begin{aligned} x + iy &= s^{(-\frac{1}{2}+i\nu_m)} \; (C_m z^{1+m} + C_{-m} z^{1-m}) \\ x - iy &= s^{(-\frac{1}{2}+i\nu_m)} \; (D_m z^{-1+m} + D_{-m} z^{-1-m}) \end{aligned}\right\}$$

and

$$\left.\begin{aligned} x + iy &= s^{(-\frac{1}{2}-i\nu_m)} \; (C'_m z^{1+m} + C'_{-m} z^{1-m}) \\ x - iy &= s^{(-\frac{1}{2}-i\nu_m)} \; (D'_m z^{-1+m} + D'_{-m} z^{-1-m}) \end{aligned}\right\}$$

must be added to yield

$$\left.\begin{aligned} x + iy &= s^{-\frac{1}{2}} (C_m s^{i\nu_m} z^{1+m} + C_{-m} s^{i\nu_m} z^{1-m} + C'_m s^{-i\nu_m} z^{1+m} + C'_{-m} s^{-i\nu_m} z^{1-m}) \\ x - iy &= s^{-\frac{1}{2}} (D_m s^{i\nu_m} z^{-1+m} + D_{-m} s^{i\nu_m} z^{-1-m} + D'_m s^{-i\nu_m} z^{-1+m} + D'_{-m} s^{-i\nu_m} z^{-1-m}) \end{aligned}\right\} \tag{25}$$

From $x + iy = \overline{x - iy}$ follows this time

$$\begin{aligned} C_m &= \overline{D'_{-m}}, & C'_m &= \overline{D_{-m}}, \\ C_{-m} &= \overline{D'_m}, & C'_{-m} &= \overline{D_m}. \end{aligned}$$

Substitution into (17) leads to

$$\begin{aligned} \overline{C'_{-m}} &= \rho\left[1 + (2\lambda_m + 2)/(m-1)\right] C_m, \\ \overline{C_{-m}} &= \rho\left[1 + (2\lambda'_m + 2)/(m-1)\right] C'_m. \end{aligned} \tag{26}$$

The particular solutions can be combined linearly to yield a range of functions, satisfying the differential equations, and any boundary condition, within certain limits.

4. *Boundary Values along a Closed Contour.*

The next step to be performed is to decide which complete solution fulfills the given boundary conditions; in other words, the coefficients C_m must be computed from the boundary values. Let the stresses be given along a closed contour Γ, e.g. the perimeter of a hole. From these given stresses s and θ can be computed at every point of Γ. Two solutions are found, more commonly known as the passive and the active solution (Rankine), or the weak and the strong (Prager). One of the two solutions is to be chosen.

If a parameter t is defined along Γ, then at any point $x_0(t) + iy_0(t)$ of Γ, s(t) and $\theta(t)$ are known.

The analysis of the foregoing sections only makes sense if x and y can be written as continuous functions of s and θ. It is therefore not only necessary that s(t) and $\theta(t)$ should be continuous and that $s(1) = s(0)$ and $\theta(1) = \theta(0) + 2k\pi$ (k being an integer number), but also that every value of θ between 0 and 2π is taken only once on the contour Γ. This means that $\theta(t)$ must be a monotonically increasing or decreasing function of t, and also that $|\theta(1) - \theta(0)|$ cannot exceed 2π. It follows that $k = \pm 1$; we choose the case $k = +1$.

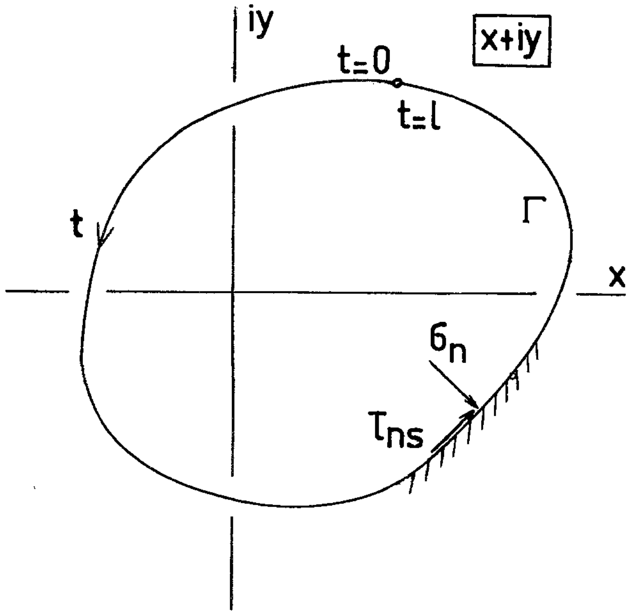

Fig. 4. The boundary contour Γ.

Th functions x_0 and y_0 then are periodic in θ, with period 2π; so along Γ we can express x_0 and y_0 by means of a Fourier series, or, which is essentially the same, $x_0 + iy_0$ by means of a Laurent series:

$$x_0(t) + iy_0(t) = \sum_{m=-\infty}^{+\infty} A_m e^{i(m+1)\theta(t)} \tag{27}$$

The problem is then solved, when one can determine the coefficients C_m and C'_m of the complete solution

$$x + iy = \sum_{m=-\infty}^{+\infty} (C_m s^{\lambda_m} + C'_m s^{\lambda'_m})e^{i(m+1)\theta} \tag{19}$$

in terms of the known coefficients A_m.

On the contour Γ the variable s is also a function of the parameter t. We will study the simpler case where s = const along Γ; by an appropriate choice of the constant σ_0 (7) s can be made equal to unity. One has now to compute C_m and C'_m from the relation

$$\sum_{m=-\infty}^{+\infty} A_m e^{i(m+1)\theta} = \sum_{m=-\infty}^{+\infty} (C_m + C'_m)e^{i(m+1)\theta} ,$$

or $$C_m + C'_m = A_m . \tag{28}$$

From (20) and (21) it is known that C'_0 is real and C_0 is imaginary, therefore

$$\begin{aligned} C_0 &= i\,\mathrm{Im}\{A_0\} , \\ C'_0 &= \mathrm{Re}\{A_0\} . \end{aligned} \tag{29}$$

From (22) it is known that $C_1 = 0$

so $$\begin{aligned} C'_1 &= A_1 , \\ C'_{-1} &= \rho^{-1}\bar{A}_1 , \\ C_{-1} &= A_{-1} - \rho^{-1}\bar{A}_1 . \end{aligned} \tag{30}$$

For m = 2, 3,... one has the relations

$$C_m + C'_m = A_m ,$$
$$C_{-m} + C'_{-m} = A_{-m} .$$

The second of these can, with help of (26), be transformed into a second relation for C_m and C'_m and remembering that $\lambda_m = -\frac{1}{2} + i\nu_m$:

$$\rho[1 + (1 + 2i\nu_m)/(m-1)]\,C_m + \rho[1 + (1 - 2i\nu_m)/(m-1)]\,C'_m = \bar{A}_{-m} .$$

Solving for C_m and C'_m yields

$$\begin{aligned} C_m &= [\tfrac{1}{2} + i\tfrac{1}{4}m/\nu_m]A_m - [i(m-1)/4\rho\nu_m]\overline{\bar{A}_{-m}} , \\ C'_m &= [\tfrac{1}{2} - i\tfrac{1}{4}m/\nu_m]A_m + [i(m-1)/4\rho\nu_m]\overline{\bar{A}_{-m}} . \end{aligned} \tag{31}$$

Equation (31) holds for m = +2, +3,... as well as for m = -2, -3,.... This analysis shows how, from a given shape of the hole and from the given stresses along the perimeter of the hole, the solution of the stresses in the plastic region can be performed. The quantity x + iy is found as a function of s and θ; x and y are its real and imaginary parts. An example of such a function will be dealt with in sec. 6.

5. *Some Properties of the Solution.*

i. The special case where $\tau_{ns} = 0$, σ_n = const along the boundary.

Let Γ be the perimeter of a hole; let the external force, exerted from inside Γ onto the material outside Γ, be a constant normal pressure. This implies that the shear stress along Γ is zero, so the directions of the principal stresses are the tangent and the normal to Γ.

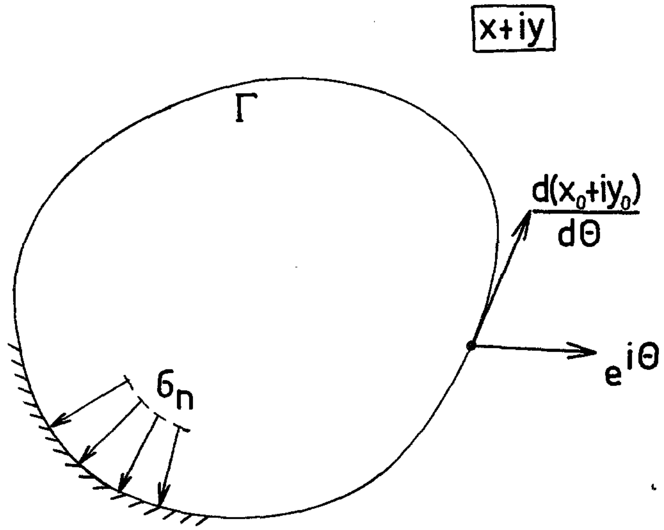

Fig. 5.

The complex number

$$\frac{d(x_0 + iy_0)}{d\theta} = \sum_{m=-\infty}^{+\infty} i(m+1)A_m e^{i(m+1)\theta} \tag{32}$$

has the direction of the tangent; if the major principal stress, which has the direction θ, is tangential to Γ, then

$$\arg \left[\frac{d(x_0 + iy_0)}{d\theta}\right] = \theta.$$

Hence in this case, the factor $\sum_{m=-\infty}^{+\infty} i(m+1)A_m e^{im\theta}$ must be real for every value of θ; so that

$$i(m+1)A_m = \overline{i(-m+1)A_{-m}}$$

or $\quad (m+1)A_m = (m-1)\overline{A_{-m}}, \quad m = 1, 2, 3, \ldots$.

and $\quad \mathrm{Re}\{A_0\} = 0 \qquad (33)$

On the other hand, $e^{i\theta}$, multiplied by a purely imaginary factor, gives rise to a complex number whose argument is $\theta + \frac{1}{2}\pi$. So, if we consider the other case, that the major principal stress is directed along the normal to Γ, then $\arg[d(x_0 + iy_0)/d\theta] = \theta \pm \frac{1}{2}\pi$; thus in this case

$$(m+1)A_m = -(m-1)\overline{A_{-m}}, \quad m = 1, 2, 3, \ldots$$
$$\mathrm{Im}\{A_0\} = 0 \qquad (34)$$

In the latter case one sees that the solution (20) does not appear. Then the solution (21) dominates, s approaches zero at a great distance from the hole, whereas, in the first of the two cases, the solution (21) is absent, so that the stresses grow without limit at a great distance.

***ii.* The lines** along which θ = const intersect the boundary curve **Γ at right angles, when** σ_n = const and $\tau_{ns} = 0$ along Γ (see fig. 6).

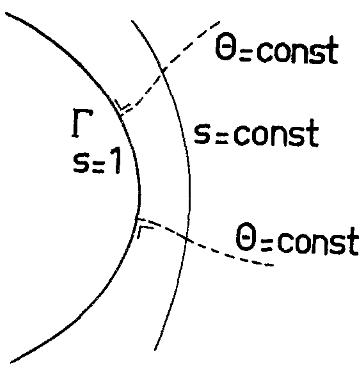

Fig. 6.

The lines θ = const are found from the total solution by differentiating partially with respect to s:

$$\frac{\partial(x+iy)}{\partial s} = \sum_{m=-\infty}^{+\infty} (\lambda_m C_m s^{\lambda_m - 1} + \lambda'_m C'_m s^{\lambda'_m - 1})e^{i(m+1)\theta}$$

Consider now the points of Γ, in other words, the points where s = 1

$$\left[\frac{\partial(x+iy)}{\partial s}\right]_{s=1} = \sum_{m=-\infty}^{+\infty} (\lambda_m C_m + \lambda'_m C'_m)e^{i(m+1)\theta} =$$
$$= e^{i\theta} \sum_{m=-\infty}^{+\infty} (-\tfrac{1}{2}A_m - \frac{m}{2}A_m + \frac{m-1}{2\rho}\overline{A_{-m}})e^{im\theta}$$

This vector is indeed perpendicular to the vector (32), if either (33) or (34) is fulfilled. This can be seen as follows.

If condition (33) is fulfilled, then

$$\left[\frac{\partial(x+iy)}{\partial s}\right]_{s=1} = e^{i\theta} \sum_{m=-\infty}^{+\infty} (m+1)A_m e^{im\theta} \left(-\tfrac{1}{2}+\tfrac{1}{2}\rho^{-1}\right) = -i\lambda_0 \frac{\partial(x+iy)}{\partial\theta}$$

If (34) is fulfilled, then

$$\left[\frac{\partial(x+iy)}{\partial s}\right]_{s=1} = e^{i\theta} \sum_{m=-\infty}^{+\infty} (m+1)A_m e^{im\theta} \left(-\tfrac{1}{2}-\tfrac{1}{2}\rho^{-1}\right) = -i\lambda_0' \frac{\partial(x+iy)}{\partial\theta} \tag{36}$$

We see that $\partial(x+iy)/\partial s$ and $\partial(x+iy)/\partial\theta$ differ by a purely imaginary factor, so that the difference of their arguments is $\frac{1}{2}\pi$, i.e. they are perpendicular to each other.

In the factor λ_0 or λ_0' also appear, which is not surprising, since the contour Γ can locally be considered as a circular arc. One finds again λ_0 or λ_0', depending on whether one has to do with the active or the passive solution.

iii. The resultant boundary force, expressed into the coefficients C_m.

The resultant of the external forces exerted on the material outside an arbitrary contour Γ', which encloses Γ, must be the same for every such Γ', because the material was assumed to be weightless. Thus we can choose for Γ' a contour s = const.

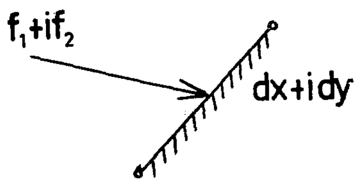

Fig. 7. The force on a small segment.

It is possible to write the force exerted on a small segment in the form of a complex number:

$$f_1 = \sigma_x dy - \tau_{xy} dx,$$

$$f_2 = -\sigma_y dx + \tau_{xy} dy,$$

$$\begin{aligned} f_1 + if_2 &= (\sigma_x + i\tau_{xy})dy + (-i\sigma_y - \tau_{xy})dx = \\ &= (s\sigma_0 - c\cot g\,\varphi)(dy - idx) + \rho s\sigma_0 e^{2i\theta}(dy + idx) = \\ &= -i(s\sigma_0 - \cot g\,\varphi)(dx + idy) + i\rho s\sigma_0 e^{2i\theta}(dx - idy). \end{aligned} \tag{37}$$

Both terms are integrated along Γ' separately:

$$-i(s\sigma_0 - c\cot g\,\varphi)\oint_{\Gamma'}(dx + idy) = 0,$$

$$i\rho s\sigma_0 \oint_{\Gamma'} e^{2i\theta}(dx - idy) =$$

$$= i\rho s\sigma_0 \sum_{m=-\infty}^{+\infty}\int_0^{2\pi} e^{2i\theta}\,\overline{(C_m s^{\lambda_m} + C_m' s^{\lambda_m'})}\,(-i)(m+1)e^{-i(m+1)\theta}\,d\theta.$$

The integral of every term vanishes, except the term m = +1; so there remains

$$\rho s\sigma_0 . 2\pi . 2\,(C_1 s^{\lambda_1} + C_1' s^{\lambda_1'}),$$

according to (22): $C_1 = 0$,

according to (23): $\lambda_1' = -1$,

so that the result is independent of s, as is necessary.
The resultant force is thus:

$$4\pi\rho\sigma_0 C_1' . \tag{38}$$

iv. The resultant moment of the boundary forces.

The moment of the force (37) on a small segment dx + idy, taken with respect to the origin is:

$$xf_2 - yf_1 = \mathrm{Im}\,[(x - iy)\,(f_1 + if_2)] =$$
$$= \mathrm{Im}\,[(x - iy)\,\{-i(s\sigma_0 - c\,\mathrm{cotg}\,\varphi)\,(dx + idy) + i\rho s\sigma_0 e^{2i\theta}\,(dx - idy)\}] . \tag{39}$$

Again both terms are integrated along a closed contour Γ', where s = const.

$$\oint_{\Gamma'} \mathrm{Im}\,[(x - iy)\,\{-i(s\sigma_0 - c\,\mathrm{cotg}\,\varphi)\,(dx + idy)\}] =$$
$$= (s\sigma_0 - c\,\mathrm{cotg}\,\varphi)\mathrm{Im}\left[\int_0^{2\pi} \sum_{m=-\infty}^{+\infty} \overline{(C_m s^{\lambda_m} + C_m' s^{\lambda_m'})e^{-i(m+1)\theta}}\,(-i)\,\times\right.$$
$$\left.\times \sum_{m=-\infty}^{+\infty} (C_k s^{\lambda_k} + C_k' s^{\lambda_k'})i(k+1)e^{i(k+1)\theta}\right] d\theta =$$
$$= (s\sigma_0 - c\,\mathrm{coth}\,\varphi)\mathrm{Im}\,2\pi\left[\sum_{m=-\infty}^{+\infty} \overline{(C_m s^{\lambda_m} + C_m' s^{\lambda_m'})}\,(C_m s^{\lambda_m} + C_m' s^{\lambda_m'})\,(m+1)\right] = 0,$$

because every term of the sum between square brackets is real.
Integration of the second term yields

$$\oint_{\Gamma'} \mathrm{Im}\left[i\rho s\sigma_0\, e^{2i\theta}\,(x - iy)\,d(x - iy)\right] =$$
$$= \rho s\sigma_0\ \mathrm{Im}\left[\int_0^{2\pi} e^{2i\theta} \sum_{m=-\infty}^{+\infty} \overline{(C_m s^{\lambda_m} + C_m' s^{\lambda_m'})e^{-i(m+1)\theta}}\ \times\right.$$
$$\left.\sum_{m=-\infty}^{+\infty} \overline{(C_k s^{\lambda_k} + C_k' s^{\lambda_k'})\,(k+1)e^{-i(k+1)\theta}}\ \ d\theta\right] =$$
$$= 2\pi\rho s\sigma_0\ \mathrm{Im}\left[\sum_{m=-\infty}^{+\infty} (1-m)\ \overline{(C_m s^{\lambda_m} + C_m' s^{\lambda_m'})\,(C_{-m} s^{\lambda_m} + C_{-m}' s^{\lambda_m'})}\right] =$$
$$= 2\pi\rho\ \sigma_0\ \mathrm{Im}\left[\sum_{m=-\infty}^{+\infty} (1-m)\ \overline{(C_m C_{-m}' + C_{-m} C_m' + C_m C_{-m} s^{2\lambda_m+1} + C_m' C_{-m}' s^{2\lambda_m'+1}\,)}\right];$$

$C_m C_{-m}\, s^{2\lambda_m+1} + C_m' C_{-m}'\, s^{2\lambda_m'+1}$ is real and therefore its imaginary part vanishes, because

C_0 is imaginary, λ_0 is real;

C_0' is real, λ_0' is real;

$C_1 C_{-1} = 0$, because $C_1 = 0$;

$C_1' C_{-1}' = \overline{C_1' C_{-1}'}$, as is seen in (23), λ_1' is real;

$C_m' C_{-m}' = \overline{C_m C_{-m}}$ follows from (26), m = 2, 3, . . . ,

$s^{2\lambda_m+1} = \overline{s^{2\lambda_m+1}}$, because $\lambda_m = \overline{\lambda_m'}$.

The resultant moment is thus

$$2\pi\rho\sigma_0 \sum_{m=-\infty}^{+\infty} (1-m)\,\mathrm{Im}\left[\overline{C_m C'_{-m} + C_{-m} C'_m}\right] =$$

$$= -2\pi\rho\sigma_0 \left\{\sum_{m=-\infty}^{+\infty} (1-m)\,\mathrm{Im}\left[C_m C'_{-m}\right] + \sum_{\mu=-\infty}^{+\infty} (1-\mu)\,\mathrm{Im}\left[C_{-\mu} C'_{\mu}\right]\right\} =$$

$$= -2\pi\rho\sigma_0 \left\{\sum_{m=-\infty}^{+\infty} (1-m)\,\mathrm{Im}\left[C_m C'_{-m}\right] + \sum_{m=-\infty}^{+\infty} (1+m)\,\mathrm{Im}\left[C_m C'_{-m}\right]\right\} =$$

$$= -4\pi\rho\sigma_0 \sum_{m=-\infty}^{+\infty} \mathrm{Im}\left[C_m C'_{-m}\right]. \tag{40}$$

6. *Two Examples*.

i. In the first example Γ is an oval, loaded with a constant normal pressure. Its coefficients are taken

$$\begin{aligned} A_0 &= i, \\ A_2 &= 0.1\, i, \\ A_{-2} &= -0.3\, i. \end{aligned} \tag{41}$$

These values satisfy condition (33). From (29) and (31) the coefficients of the total solution follow:

$$\begin{aligned} C_0 &= i, \\ C_2 &= \tfrac{1}{2}(1 + i\nu_2^{-1})A_2 - i\overline{A_{-2}}/(4\rho\nu_2), \\ C_{-2} &= \tfrac{1}{2}(1 - i\nu_2^{-1})A_{-2} + 3i\,\overline{A_2}/(4\rho\nu_2), \end{aligned}$$

According to (19) the total solution is, if we write $\nu_2 \ln s = \zeta$,

$$\begin{aligned} x + iy = {} & i\, s^{(1-\rho)/2\rho}\, e^{i\theta} \\ & + 0.1\, i\, s^{-\frac{1}{2}} \left\{(\cos\zeta + (3-2\rho)\sin\zeta/4\rho\nu_2)e^{3i\theta} + \right. \\ & \left. + (-3\cos\zeta + (3-6\rho)\sin\zeta/4\rho\nu_2)e^{-i\theta}\right\}. \end{aligned} \tag{42}$$

This function is shown in fig. 8, where the angle of internal friction φ is

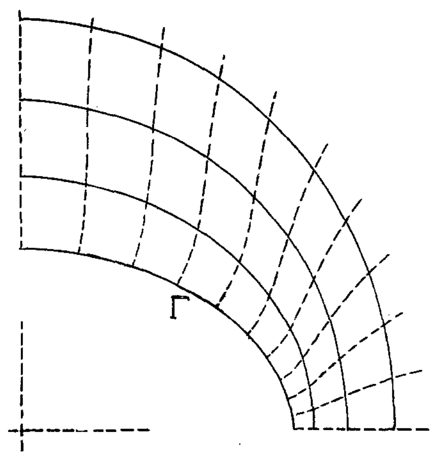

Fig. 8. One quadrant of the physical plane.

chosen as $\pi/6$. It represents the distribution of the stresses in the plastic zone around a hole of oval shape, which is loaded with a constant normal pressure and where the major principal stress is tangential to the boundary. In fig. 8 the lines of constant s (constant isotropic stress) are indicated by full lines, the lines of constant θ by broken lines.

ii. A circular hole loaded by a normal pressure and a shear stress.

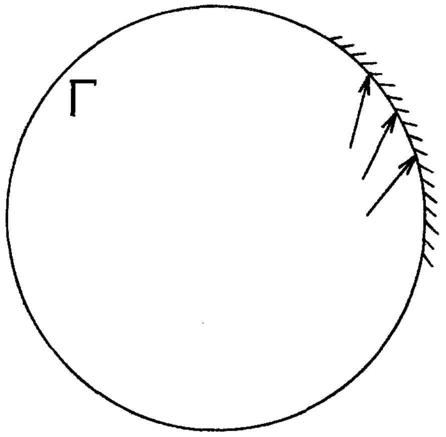

Fig. 9. A circular hole loaded by a normal pressure and shear stress.

In this case we have only to do with both circular solutions (20) and (21). When (20) and (21) are added, one gets

$$x + iy = C_o\, s^{\lambda_o}\, e^{i\theta} + C_o' s^{\lambda_o'}\, e^{i\theta} ,$$

where C_o' is real and C_o is imaginary, so that these coefficients can be written as

$$C_o = i \sin \psi,$$
$$C_o' = \cos \psi,$$

if the radius of Γ is taken to be unity. Along the boundary Γ, s again equals 1, so that

$$x_o + iy_o = e^{i(\theta+\psi)} .$$

It can be seen from (40) that the resultant moment of the boundary forces is $-4\pi\rho\sigma_o \mathrm{Im}\,[C_o C_o'] = -4\pi\rho\,\sigma_o C_o'(-iC_o)$.

Indeed, if either C_o or C_o' vanishes, then the moment also disappears, but in the above case the resulting moment is $-2\pi\rho\sigma \sin 2\psi$. This is only natural since the shear stress along Γ is $\tau_{ns} = -\rho\sigma_o \sin 2\psi$. It is uncertain whether s approaches zero or infinity for great values of x and y. To settle this problem we must look at the behaviour of the solution x + iy in the neighbourhood of Γ. The radial coordinate $r = [x^2 + y^2]^{\frac{1}{2}}$ may not become smaller than 1, for this would be impossible physically.

$$r = \left[|C_o|^2 s^{2\lambda_o} + |C_o'|^2 s^{2\lambda_o'}\right]^{\frac{1}{2}} ,$$
$$\frac{dr}{ds} = \frac{1}{r}\,(\lambda_o |C_o|^2 s^{2\lambda_o - 1} + \lambda_o' |C_o'|^2 s^{2\lambda_o' - 1});$$

on Γ: $\dfrac{dr}{ds} = \lambda_o |C_o|^2 + \lambda_o' |C_o'|^2 .$

If $dr/ds < 0$ for $s = 1$, then s must decay to ensure that r grows from unity. Then s must vanish for great values of r. In the other case, when $[dr/ds]_{s=1} > 0$, s will approach infinity.

This second example can be shown to be identical with an example given by Nadai [6].

7. *Conclusion.*

In the foregoing sections a series of solutions of the problem of plastic plane strain have been found, all of which are of the following form: the Cartesian coordinates x and y of the physical plane are trigonometrical functions of θ, the direction of the major principal stress, multiplied by a power of s, a quantity directly connected with the isotropic stress. If the boundary condition can be described in the same form, the boundary value problem can be solved. In sec. 4 this was done for a special sort of boundary condition. There the shape of the boundary was arbitrary, but the surface traction was a constant normal pressure.

The analytical method seems very suitable for the determination of the stresses in the plastic region around a hole. The method may also be applicable to other sorts of plasticity problems, but this is beyond the scope of the present paper.

The possibilities are limited in the first place by the requirement that along the boundary the functions θ and s are continuous and further that there exists a one-to-one relation between the points (θ, s) and (x, y) of the boundary.

One conclusion of practical importance to be drawn from sec. 3 is that the plastic stress distribution around a hole of general shape, and loaded by an arbitrary surface traction, will tend to circular symmetry at a great distance from the hole. An example of this behaviour is shown in fig. 8.

Acknowledgement.

I wish to thank Prof. Dr. R. Timman, Prof. Dr. Ir. G. de Josselin de Jong and Dr. R. E. Gibson for their kind and valuable help preparing the paper.

References.

1. Nottrot, R. and R. Timman — A general method of solving the plane elasto-plastic problem. Journal of Engineering Mathematics, Vol. I, 1, jan. 1967, p. 19-36.
2. Sokolovsky, V.V. — A plane problem of the theory of plasticity on the distribution of stresses around a hole (Russ.) Prikl. Mat. i Mekh. XIII, 1949, 159-164.
3. Sokolovsky, V.V. — Approximate integration of the equations of a plane problem of the theory of plasticity (Russ.) Prikl. Mat. i Mekh. XIII, 1949, 321-322.
4. Geiringer, H. — Recent results in the theory of plasticity. Adv. Appl. Mech. III, 1953, 197-294.
5. Terzaghi, K. — Theoretical Soil Mechanics. J. Wiley, New York/London, 1959 (9th pr.)
6. Nadai, A. — Ueber die Gleit- und Verzweigungsflächen einiger Gleichgewichtszustände bildsamer Massen und die Nachspannungen bleibend verzerrter Körper. Zeitschrift für Physik 30, 1924, 106-138.

[Received January 30, 1967]